Collisional effects on diffusion scaling laws in electrostatic turbulence

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(Received 28 June 1999)

The effect of particle collisions on effective transport in two-dimensional electrostatic plasma turbulence is studied analytically in the framework of a test particle approach. We show that an amplification of the diffusion coefficient can be produced by the combined effect of collisions and trajectory trapping in the structure of the stochastic potential.

PACS number(s): 52.35.Ra, 52.25.Fi, 02.50.−r, 05.40.−a

I. INTRODUCTION

Particle and energy transport in turbulent plasmas has been intensively studied in the context of plasma fusion research. In addition to self-consistent analyses which deal with plasma instabilities and their nonlinear development, the test particle approach leads to interesting results. Detailed studies in this framework were devoted to the influence of particle collisions on the effective transport in magnetic turbulence [1–4]. They proved the existence of several transport regimes which are rather different from the collisionless one. However, a similar study concerning the diffusion of collisional particles in electrostatic turbulence has not been performed until now, to our knowledge. This problem is treated here. More precisely, we determine the time dependent diffusion coefficient of collisional particles moving in a two-dimensional stochastic potential as generated in a magnetically confined (tokamak) plasma.

Previous papers [5–7] demonstrated that a process of trajectory trapping around the extrema of the potential appears in a low frequency turbulence. This process determines a decrease of the diffusion coefficient and the change of its scaling in the parameters of the stochastic field (represented by the Kubo number K which will be defined in Sec. II). More precisely, this trapping process changes the dependence on K of the diffusion coefficient from a Bohm to a percolation scaling law [8]. We show here that a weak collisional diffusivity of the particles can produce a decisive influence on the effective diffusion. Important nonlinear effects are shown to appear in close connection with trajectory trapping. A detailed study of the possible diffusion regimes is presented. The dependence of the diffusion coefficient on the Kubo number can be significantly changed due to collisions.

Our model is based on Langevin-type equations that describe particle guiding center trajectories. We use the *decorrelation trajectory method* [9], a recently developed statistical approach describing the complicated process of trajectory trapping.

The problem of the evolution of collisional particles in stochastic velocity fields was studied for the case of frozen turbulence (quenched disorder or ''random random walks'') in a large number of papers related especially to solid state physics and fluid turbulence. A detailed review of this work was presented in Refs. [10] and [8]. See also the very recent review paper by Majda and Kramer [11]. The main aim of these studies was to find under what conditions the process is superdiffusive or subdiffusive, and to determine the asymptotic exponent γ of the time dependence of the mean square displacement of the particles ($\overline{R^2(t)} \sim t^{2\gamma}$). From the mathematical point of view, these studies use renormalization group techniques which are adequate for determining γ but not the details of the evolution of the stochastic motion which are contained in the running diffusion coefficient or in the Lagrangian correlation of the velocity. Although our model is different from that of Ref. [10] in the sense that we are interested in time dependent stochastic fields as they appear in plasma turbulence, we show that our results are in qualitative agreement with those of Ref. [10]. This strengthens the idea that the decorrelation trajectory method is a powerful tool for analyses of these kinds of problems. It has the advantage of yielding more detailed information (the Lagrangian correlation of the stochastic velocity) and also a more intuitive physical picture.

The paper is organized as follows. The model and the system of equations are formulated in Sec. II. A short description of the process of trajectory trapping around the extrema of the stochastic potential and of the decorrelation trajectory method is presented in Sec. III. The effect of particle collisions is treated in Sec. IV, where the running diffusion coefficient is determined. Section V contains the analyses of the results, and Sec. VI a detailed study of the possible diffusion regimes. The conclusions are summarized in Sec. VII.

II. MODEL

The test particle models determine the transport induced by a stochastic field whose statistical characteristics are supposed to be known from experimental measurements. This allows one to ignore the collective effects which produce the stochastic field, and to concentrate on determining the statistics of the individual particle trajectories. In the case of Gaussian stochastic fields, a complete experimental description consists of measuring the average velocity and the two-point Eulerian correlation of the potential (which is equivalent to the wave number spectrum). Two equivalent

mathematical formulations are possible for these models: the Langevin equation for the stochastic trajectories or the Fokker-Planck equation for the evolution of the probability density of the displacements. The first variant is used in this paper.

We consider a slab geometry with a strong confining magnetic field B_0 along the z axis and a fluctuating potential $\phi(\mathbf{x},t)$ in the plane $\mathbf{x}=(x,y)$. The particle motion in the guiding center approximation reads.

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{v}(\mathbf{x}(t),t)+\boldsymbol{\eta}(t), \quad \mathbf{x}(0)=\mathbf{0}, \tag{1}$$

where

$$\mathbf{v}(\mathbf{x},t)=-\frac{\nabla\phi(\mathbf{x},t)\times\mathbf{e}_z}{B_0}, \tag{2}$$

and $\boldsymbol{\eta}(t)$ is the collisional stochastic velocity. The electrostatic potential $\phi(\mathbf{x},t)$ is a stochastic field considered to be Gaussian, stationary, and homogeneous. Since the velocity components are derivatives of the potential, they are Gaussian, stationary, and homogeneous as well. We assume that they have zero averages. The two-point Eulerian correlation (EC) function of the potential is modeled by

$$E(\mathbf{x},t)\equiv\langle\phi(\mathbf{x}_1,t_1)\phi(\mathbf{x}_1+\mathbf{x},t_1+t)\rangle = \beta^2\mathcal{E}(\mathbf{x})\exp\left(-\frac{|t|}{\tau_c}\right). \tag{3}$$

Angular brackets denote the statistical average over the realizations of the stochastic potential field, β is the amplitude of the potential fluctuations (divided by B_0), and τ_c is their correlation time. $\mathcal{E}(\mathbf{x})$ is a dimensionless function which decays from $\mathcal{E}(\mathbf{0})=1$ (where it has a maximum) to zero when $|\mathbf{x}|\to\infty$; its form is left unspecified at this stage. A dimensionless parameter, *the Kubo number* can be defined as

$$K=V\tau_c/\lambda, \quad V=\beta/\lambda, \tag{4}$$

where V measures the amplitude of the fluctuating velocity and λ is a characteristic wavelength of the turbulence, which will be called the correlation length. The Kubo number is thus the ratio of the average distance covered by the particles during the correlation time of the stochastic velocity field to its correlation length. It represents a measure of the particle's capacity of exploring the space structure of the velocity field before the latter changes. In mathematical terms, this Kubo number determines the importance of the Lagrangian nonlinearity introduced in Eq. (1) by the space dependence of the velocity field.

The two-point EC of the velocity components and of the potential with the velocity, $E_{ij}(\mathbf{x},t)\equiv\langle v_i(\mathbf{0},0)v_j(\mathbf{x},t)\rangle$ and $E_{\phi i}\equiv\langle\phi(\mathbf{0},0)v_i(\mathbf{x},t)\rangle$, are obtained from $E(\mathbf{x},t)$ by appropriate derivatives:

$$E_{xx}=-\frac{\partial^2}{\partial y^2}E, \quad E_{yy}=-\frac{\partial^2}{\partial x^2}E, \quad E_{xy}=\frac{\partial^2}{\partial x\partial y}E,$$

$$E_{x\phi}=-E_{\phi x}=\frac{\partial}{\partial y}E, \quad E_{y\phi}=-E_{\phi y}=-\frac{\partial}{\partial x}E. \tag{5}$$

The collisional velocity $\boldsymbol{\eta}$ is modeled by a zero average white Gaussian noise,

$$\overline{\eta_i(t_1)\eta_j(t_1+t)}=\delta_{ij}\chi\nu\delta(\nu|t|), \tag{6}$$

where ν is interpreted as the collision frequency of the plasma, and $\chi=V_{th}^2\nu/2\Omega^2$ as the (classical) cross-field collisional diffusion coefficient (V_{th} is the thermal velocity and $\Omega=eB_0/m$ is the gyration frequency). The bar represents the average over the realizations of the collisional velocity η. The collisions introduce a second dimensionless parameter, the Péclet number [8]

$$P=\frac{\beta}{\chi}=\frac{\tau_{coll}}{\tau_{tr}}, \tag{7}$$

where the transit time $\tau_{tr}=\lambda/V$ is the time during which a particle with the velocity V goes through the correlation length, and the collisional time $\tau_{coll}=\lambda^2/\chi$ is the time during which the collisional mean square displacement attains λ^2. The Péclet number is a measure of the particle's capacity of exploring the space structure of the velocity field in the presence of collisions which scatter the trajectories. Since Eq. (4) can be written as $K=\tau_c/\tau_{tr}$, one can note that Kubo and Péclet numbers are similar in the sense that both describe physical effects (time variation of the potential and collisions respectively) which perturb the motion along the potential contour lines. Large values of P (i.e., a small collisional diffusion χ) and large values of K correspond to nonlinear regimes strongly influenced by the structure of the stochastic potential.

The mean square displacement (MSD) of the particles and the running diffusion coefficient are determined from the two-point correlation function of the Lagrangian velocity. The latter is defined as

$$L_{ij}(t;t_1)\equiv\overline{\langle v_i(\mathbf{x}(t_1),t_1)v_j(\mathbf{x}(t_1+t),t_1+t)\rangle}, \tag{8}$$

and will be called for simplicity *Lagrangian velocity correlation* (LVC). In the stationary and homogeneous case one can assume that the LVC depends only on the time interval t, and the MSD can be written as

$$\overline{\langle x_i^2(t)\rangle}=2\int_0^t d\tau L_{ii}(\tau)(t-\tau). \tag{9}$$

The running diffusion coefficient defined as $D_i(t)\equiv\frac{1}{2}(d/dt)\overline{\langle x_i^2(t)\rangle}$ is

$$D_i(t)=\int_0^t d\tau L_{ii}(\tau). \tag{10}$$

Thus the diffusion problem reduces to the determination of the LVC, knowing the statistical description of the stochastic potential. We use the decorrelation trajectory method which was developed for the collisionless problem in Ref. [9]. A short review of these results and of the method is presented in Sec. III. This is necessary for understanding the effect introduced by particle collisions. The latter is determined in Sec. IV, where the running diffusion coefficient is determined for all values of the Kubo and Péclet numbers.

III. PARTICLE TRAPPING AND THE DECORRELATION TRAJECTORY METHOD

The particular case of collisionless particles [$\boldsymbol{\eta}=\mathbf{0}$ in Eq. (1), i.e., $P\rightarrow\infty$] was studied earlier both numerically by means of direct simulation of trajectories and analytically. For small Kubo numbers (quasilinear regime), the results are well established: the diffusion coefficient is $D_{QL}=(\lambda^2/\tau_c)K^2$. At large K the time variation of the stochastic potential is slow and the trajectories can approximately follow the contour lines of $\phi(\mathbf{x},t)$. The space structure of the stochastic potential has an important influence on particle trajectories. This produces a trapping effect: the trajectories are confined for long periods in small regions. A typical trajectory shows an alternation of large displacements and trapping events. The latter appear when the particles are close to the maxima or minima of the potential and consists of trajectory winding (for many turns) on almost closed small size paths. The large displacements are produced when the trajectories are at small absolute values of the potential. Numerical calculations show that the Lagrangian stochastic velocity $\mathbf{v}(\mathbf{x}(t),t)$ of the particles is Gaussian and stationary at any time during their motion. The displacements $\mathbf{x}(t)$ are Gaussian only for a very small time interval. At later times, the process of trajectory trapping determines the modification of the probability density for the displacements which develops a narrow maximum in $\mathbf{x}=\mathbf{0}$ and long tails [7]. The most important effect of trajectory trapping consists of the decrease of the diffusion coefficient and of the change of its dependence on the Kubo number from the Bohm scaling $[D_B\sim(\lambda^2/\tau_c)K]$ to percolation type scaling $[D_p\sim(\lambda^2/\tau_c)K^{0.7}]$ [5–7].

A new statistical approach, the decorrelation trajectory method, which is able to describe this complicated self-consistent trapping was developed in Ref. [9]. It applies to Gaussian stochastic fields which are homogeneous, stationary and determine stationary LVC's. Actually there are several variants of the method which lead to similar results. For reasons which will become evident later in the text, here we use the space-time decorrelation presented in the Appendix of Ref. [9].

The essential point of the new method is that it finds a set of deterministic trajectories which are determined by the EC of the potential; the LVC is then approximated using the average velocity along these trajectories. The idea is to divide the space of realizations of the stochastic potential into subensembles characterized by given values of the potential and of the velocity at the starting point of the trajectories $\mathbf{x}=\mathbf{0}$, and $t=0$:

$$\phi(\mathbf{0},0)=\phi^0,\quad \mathbf{v}(\mathbf{0},0)=\mathbf{v}^0. \tag{11}$$

The Eulerian correlation of the velocity components $E_{ij}(\mathbf{x},t)$ can be decomposed into a weighted sum of the Eulerian correlations of the velocity in each subensemble,

$$E_{ij}(\mathbf{x},t)=\int\int d\phi^0 d\mathbf{v}^0 P_1^{\phi}(\phi^0)P_1^{\phi}(\mathbf{v}^0)E_{ij}^s(\mathbf{x},t;\phi^0,\mathbf{v}^0), \tag{12}$$

where $E_{ij}^s(\mathbf{x},t;\phi^0,\mathbf{v}^0)\equiv\langle v_i(\mathbf{0},0)\,v_j(\mathbf{x},t)\rangle|_{\phi^0,\mathbf{v}^0}$ is the subensemble Eulerian correlation, i.e., it is an average conditioned by Eq. (11). $P_1^{\phi}(\phi^0)$ and $P_1^{\phi}(\mathbf{v}^0)$, respectively, are the Gaussian probability densities of the potential and of the velocity. The upperscript ϕ shows that the stochastic function in this probability is the potential ϕ. We note that Eq. (12) is an exact equation. The Eulerian correlation in the subensemble can be written as $E_{ij}^s(\mathbf{x},t;\phi^0,\mathbf{v}^0)=v_i^0\langle v_j(\mathbf{x},t)\rangle|_{\phi^0,\mathbf{v}^0}$, where $\langle v_j(\mathbf{x},t)\rangle|_{\phi^0,\mathbf{v}^0}$ is the Eulerian average velocity in the subensemble (11). The latter is determined using the Gaussian conditional probability density for having the velocity $\mathbf{v}$ in the point $(\mathbf{x},t)$ when condition (11) is imposed. Straightforward calculations lead to

$$V_j(\mathbf{x},t;\phi^0,\mathbf{v}^0)\equiv\langle v_j(\mathbf{x},t)\rangle|_{\phi^0,\mathbf{v}^0}=v_i^0\frac{E_{ij}(\mathbf{x},t)}{V^2}+\phi^0\frac{E_{\phi j}(\mathbf{x},t)}{V\beta}. \tag{13}$$

Equation (13) exhibits the space-time structure of the correlated zone. The average velocity in the subensemble (11) is $\mathbf{v}^0$ in $\mathbf{x}=0$ and $t=0$ [because $\mathcal{E}(\mathbf{x})$ has a maximum there] and it decays progressively to zero as the time and/or the distance grows. Both time and distance determine the decorrelation of the velocity.

We determine the dynamics induced by this structure by solving the (deterministic) equation

$$\frac{d\mathbf{X}(t)}{dt}=\mathbf{V}(\mathbf{X}(t),t;\phi^0,\mathbf{v}^0),\quad \mathbf{X}(0)=\mathbf{0}, \tag{14}$$

the solution of which $\mathbf{X}(t;\phi^0,\mathbf{v}^0)$ defines *the decorrelation trajectory* in the subensemble (11). We note that $\mathbf{X}(t;\phi^0,\mathbf{v}^0)$ is not an approximation of the average particle trajectory in the subensemble: rather it is a deterministic trajectory which is introduced to represent the dynamics of the decorrelation. Depending on the EC and on the parameters ϕ^0 and $\mathbf{v}^0$, two types of trajectories can be obtained in general.

(1) Trajectories along which the velocity goes to zero as $t\rightarrow\infty$. This means that the velocity eventually decorrelates from $\mathbf{v}^0$. Consequently $\mathbf{X}(t)$ saturates at a value $\mathbf{X}_d$. The mechanism of decorrelation is also exhibited by $\mathbf{X}(t)$. In this case, the decorrelation can be of two types: time decorrelation (when $\mathbf{X}_d$ is smaller than the correlation length) or space decorrelation [when the saturation of $\mathbf{X}(t)$ is produced in a time τ_d smaller than the correlation time of the stochastic field].

(2) Closed periodic trajectories. These trajectories are confined in the correlated zone: they describe the trapping in the structure of the stochastic potential.

In the present case, from Eqs. (14) and (13) we obtain closed paths for all values of ϕ^0 except $\phi^0=0$, when the path is a straight line. The size of the decorrelation paths depends on ϕ^0: it is infinite at $\phi^0=0$ and as $|\phi^0|$ grows it decreases continuously and vanishes asymptotically. The motion on these paths depends on the value of $\mathbf{v}^0$ and on the Kubo number. When $K\ll 1$ the trajectories stop after covering a small part of the path which is proportional to v^0. At large K the trajectory wind many times around the sufficiently small paths until they stop. The interval of the values of ϕ^0 (around zero) for which the decorrelation trajectories do not perform a complete turn decreases with the increase

of the Kubo number. When $K\to\infty$, the motion is no longer attenuated, and all trajectories become periodic except one (corresponding to $\phi^0=0$). This means that the decorrelation is produced only by the time variation of the stochastic potential. Its space structure produces the trapping of trajectories (space decorrelation does not exist in this stochastic motion).

The approximation on which our model is based consists of considering that the Lagrangian correlation of the velocity components is a weighted sum of the correlations observed *along the decorrelation trajectory* in each subensemble (11):

$$L_{ij}(t)\cong\int d\phi^0 d\mathbf{v}^0 P_1^{\phi}(\phi^0)P_1^{\phi}(\mathbf{v}^0)v_i^0 V_j(\mathbf{X}(t),t;\phi^0,\mathbf{v}^0)). \tag{15}$$

Actually, the decorrelation trajectories are functions of only two independent variables: $\mathbf{X}(t;\phi^0,\mathbf{v}^0)\equiv\mathbf{X}(Kut,p)$, where dimensionless variables were used taking λ as the unit for distances, τ_c for the time, β for the potential, and $V=\beta/\lambda$ for the velocities. The parameters u and p are defined by $u\equiv|\mathbf{v}^0|$ and $p\equiv\phi^0/u$. It was shown that the statistical effect of trajectory trapping is represented by a function $F(\theta)$ which is defined as

$$F(\theta)\equiv\frac{1}{\sqrt{2\pi}}\int_0^\infty dp\,du\,u^3\exp\left(-\frac{u^2(1+p^2)}{2}\right)X(u\theta;p), \tag{16}$$

where $X(u\theta,p)$ is here the x component of the decorrelation trajectories $\mathbf{X}(t;\phi^0,\mathbf{v}^0)$ calculated for $t=u\theta$, $\phi^0=pu$ and $\mathbf{v}^0=(1,0)$ along x. The LVC and the time dependent diffusion coefficient can be expressed in terms of this function as

$$L_{ij}(t)\cong\delta_{ij}\left(\frac{\beta}{\lambda}\right)^2 F'(\theta(K,t))\exp\left(-\frac{t}{\tau_c}\right), \tag{17}$$

$$D(t;K)=\frac{\lambda^2}{\tau_c}KF(\theta(K,t)), \tag{18}$$

where F' is the derivative of F, and

$$\theta(K,t)\equiv K\left[1-\exp\left(-\frac{t}{\tau_c}\right)\right]. \tag{19}$$

The asymptotic diffusion coefficient can be written as

$$D(K)=\frac{\lambda^2}{\tau_c}KF(K)=\beta F(K). \tag{20}$$

The general shape of the function $F(\theta)$ is represented in Fig. 1. The physical significance of this function results from Eqs. (18)–(20). For $\theta=Vt/\lambda\in[0,\infty)$, F represents the time dependence of the running diffusion coefficient for the stochastic particle motion in a frozen turbulence ($\tau_c\to\infty,K\to\infty$). At a fixed value $\theta=K$, $F(K)$ is the asymptotic diffusion coefficient (normalized with β) in a time dependent turbulence with Kubo number K. As seen in Fig. 1, $F(\theta)$ has a linear part at small θ [$F(\theta)=\theta$ for $\theta\ll1$], a maximum, and then a long (algebraic) decay to zero. The linear part corresponds to the quasilinear regime ($D_{QL}\sim K^2$) and the decaying part to the large K nonlinear regime. The decay is due to the decorrelation trajectories which perform many rotations, and consequently their contribution in Eq. (16) is eliminated by an incoherent mixing in the integral. The diffusion coefficient at large K is thus determined by a small part of the trajectories which are not trapped, i.e., which perform less than one rotation during the decorrelation time. A scaling in K which is close to the percolation estimate [8] was obtained in Ref. [9].

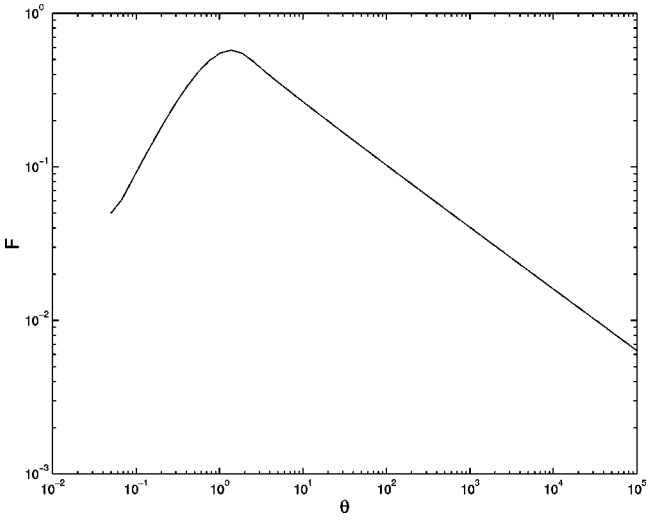

FIG. 1. The function $F(\theta)$ defined in Eq. (16) which describes the effect of trajectory trapping.

The conclusion is that the diffusion coefficient of the particles moving in a stochastic potential $\phi(\mathbf{x},t)$ results from a competition between trapping and decorrelation: the trapping process is determined by the space structure of $\phi(\mathbf{x},t)$, while the decorrelation (or trajectory release) is produced by the time variation of $\phi(\mathbf{x},t)$.

IV. PARTICLE COLLISIONS

The picture presented in Sec. III can be strongly modified by particle collisions. We come back now to the generic model presented in Sec. II. The stochastic velocity $\boldsymbol{\eta}(t)$ determines a collisional (Brownian) displacement

$$\boldsymbol{\xi}(t)=\int_0^t d\tau\,\boldsymbol{\eta}(\tau). \tag{21}$$

This equation defines a Gaussian nonstationary Markov stochastic variable known as the Wiener process [12]. This Brownian motion has zero average, and the correlation

$$\overline{\xi_i(t)\xi_j(t')}=2\delta_{ij}\chi\min[t,t']. \tag{22}$$

Since the collisions and the stochastic potential are statistically independent variables, one can perform the change of variable

$$\mathbf{x}'(t)=\mathbf{x}(t)-\boldsymbol{\xi}(t), \tag{23}$$

which transforms Eq. (1) into

$$\frac{d\mathbf{x}'(t)}{dt} = -\frac{\nabla\phi(\mathbf{x}'+\boldsymbol{\xi}(t),t)\times\mathbf{e}_z}{B_0}, \tag{24}$$

and introduces the collisional displacements in the spatial argument of the potential which becomes a doubly stochastic variable (stochastic function of a stochastic variable).

We show that the doubly stochastic field $\phi(\mathbf{x}+\boldsymbol{\xi}(t),t)$ preserves the statistical properties of the potential $\phi(\mathbf{x},t)$, namely, it is stationary, homogeneous, isotropic, and has a Gaussian one-point probability density like $\phi(\mathbf{x},t)$. The collisions determine a modification of the Eulerian correlation and of the two-point probability density of $\phi(\mathbf{x}+\boldsymbol{\xi}(t),t)$.

The probability to find the value ϕ_1 of the doubly stochastic potential in the point $(\mathbf{x},t)$ is defined by the following average over the two stochastic variables:

$$\mathcal{P}_1(\phi_1;\mathbf{x},t) \equiv \overline{\langle \delta[\phi_1 - \phi(\mathbf{x}+\boldsymbol{\xi}(t),t)\rangle]}.$$

It can be calculated using the Fourier representation of the δ function,

$$\mathcal{P}_1(\phi_1;\mathbf{x},t) = \int dq e^{-iq\phi_1} \int\int d^2\xi P_1^c(\boldsymbol{\xi},t) \times \langle \exp[iq\phi(\mathbf{x}+\boldsymbol{\xi},t)]\rangle, \tag{25}$$

where the average over the collisional noise was written explicitly using the probability density for the collisional displacements:

$$P_1^c(\boldsymbol{\xi},t) = \frac{1}{4\pi\chi t}\exp\left(-\frac{\xi^2}{4\chi t}\right). \tag{26}$$

The superscript c shows that the stochastic parameter in this probability density is the collisional velocity $\eta(t)$. Since $\phi(\mathbf{x},t)$ is Gaussian in each point $\mathbf{x}$, the remaining average in Eq. (25) is $\exp(-q^2\beta^2/2)$ and, after performing the integrals, one obtains the Gaussian probability distribution for the values of the potential:

$$\mathcal{P}_1(\phi_1;\mathbf{x},t) = \frac{1}{\sqrt{2\pi}\beta}\exp\left(-\frac{\phi_1^2}{2\beta^2}\right). \tag{27}$$

Similar straightforward calculations show that the two-point probability density for the doubly stochastic potential $\phi(\mathbf{x}+\boldsymbol{\xi}(t),t)$ is a convolution of Gaussian probabilities:

$$\mathcal{P}_2(\phi_1,\mathbf{x}_1,t_1;\phi_2,\mathbf{x}_1+\mathbf{x},t_1+t) = \int d^2\xi P_1^c(\xi,t) P_2^{\phi}(\phi_1,\mathbf{0},0;\phi_2,\mathbf{x}+\boldsymbol{\xi},t), \tag{28}$$

where P_2^{ϕ} is the two-point probability density for the potential $\phi(\mathbf{x},t)$ (in the absence of collisions).

The EC of the doubly stochastic potential $\phi(\mathbf{x}+\boldsymbol{\xi}(t),t)$, defined by

$$E^{coll} \equiv \overline{\langle \phi(\mathbf{x}_1+\boldsymbol{\xi}(t_1),t_1)\phi(\mathbf{x}_2+\boldsymbol{\xi}(t_2),t_2)\rangle}, \tag{29}$$

can be written, using Eq. (3), as

$$E^{coll} = \beta^2 \exp\left(-\frac{|t_1-t_2|}{\tau_c}\right) \times \int d^2\xi_1 d^2\xi_2 \mathcal{E}(\mathbf{x}_1+\boldsymbol{\xi}(t_1)-\mathbf{x}_2 - \boldsymbol{\xi}(t_2)) P_2^c(\boldsymbol{\xi}_1,t_1;\boldsymbol{\xi}_2,t_2), \tag{30}$$

where $P_2^c(\boldsymbol{\xi}_1,t_1;\boldsymbol{\xi}_2,t_2)$ is the two-point probability density for having the collisional displacement $\boldsymbol{\xi}_1$ at t_1 and $\boldsymbol{\xi}_2$ at t_2. For $t_2>t_1$, $P_2^c(\boldsymbol{\xi}_1,t_1;\boldsymbol{\xi}_2,t_2) = P_1^c(\boldsymbol{\xi}_1,t_1)P_1^c(\boldsymbol{\xi}_2-\boldsymbol{\xi}_1,t_2-t_1)$. Straightforward calculations lead to the following expression for the EC of the potential in the presence of collisions:

$$E^{coll}(\mathbf{x},t) = \beta^2 \mathcal{E}^{coll}(\mathbf{x},t)\exp\left(-\frac{|t|}{\tau_c}\right), \tag{31}$$

where

$$\mathcal{E}^{coll}(\mathbf{x},t) \equiv \int\int d^2\xi\, \mathcal{E}(\mathbf{x}+\boldsymbol{\xi}) P_1^c(\boldsymbol{\xi},t). \tag{32}$$

In Eqs. (31) and (32), $\mathbf{x}=\mathbf{x}_2-\mathbf{x}_1$ and $t=t_2-t_1$; this shows that the collision-averaged EC of the potential $E^{coll}(\mathbf{x},t)$ is stationary and homogeneous, like $E(\mathbf{x},t)$.

Thus the average effect of the collisional noise $\boldsymbol{\eta}(t)$ consists of the modification of the EC of the potential. More specifically, the space dependence of the correlation $\mathcal{E}(\mathbf{x})$ is transformed into $\mathcal{E}^{coll}(\mathbf{x},t)$ [Eq. (32)], gaining an additional time dependence which is a *diffusive evolution*. Indeed, $\mathcal{E}^{coll}(\mathbf{x},t)$ is the solution of the diffusion equation.

$$\frac{\partial}{\partial t}\mathcal{E}^{coll}(\mathbf{x},t) = \chi\nabla^2\mathcal{E}^{coll}(\mathbf{x},t), \tag{33}$$

with the initial condition $\mathcal{E}^{coll}(\mathbf{x},t=0)=\mathcal{E}(\mathbf{x})$. Consequently, the average effect of collisions consists of progressively smoothing out the EC of the potential and of eliminating asymptotically the space dependence of the correlation.

Particle collisions do not affect the isotropy property of the potential: if $E(\mathbf{x},t)$ is isotropic then $E^{coll}(\mathbf{x},t)$ is a function of $r\equiv|\mathbf{x}|$ as well. In these conditions a change of variable $\boldsymbol{\xi}'=\mathbf{x}+\boldsymbol{\xi}$ can be performed in Eq. (32) and the polar coordinates $\boldsymbol{\xi}'=(\rho,\alpha)$ (where α is the angle formed by the vectors $\mathbf{x}$ and $\boldsymbol{\xi}'$) can be introduced. The integral over α can be performed analytically, yielding

$$\mathcal{E}^{coll}(r,t) = \frac{1}{\overline{\xi^2(t)}}\int_0^{\infty} d\rho\,\rho\mathcal{E}(\rho) I_0\left(\frac{r\rho}{\overline{\xi^2(t)}}\right)\exp\left(-\frac{r^2+\rho^2}{2\overline{\xi^2(t)}}\right), \tag{34}$$

where I_0 is the Bessel function of imaginary argument. Equation (34) indeed shows that $\mathcal{E}^{coll}$ depends only on r when $E(\mathbf{x},t)$ is isotropic.

We note that in the correlation $E^{coll}(\mathbf{x},t)$ there is an important difference between the time dependence determined by the time variation of the potential [the exponential factor in Eq. (31)] and the time dependence induced by the collisions. The space integral of $E^{coll}(\mathbf{x},t)$ is

$$\int\int d^2x\, E^{coll}(\mathbf{x},t) = \beta^2\exp\left(-\frac{t}{\tau_c}\right)\int\int d^2x\,\mathcal{E}(\mathbf{x}),$$

where the effect of collisions has disappeared. This shows that the time variation of the potential destroys the correlations, while particle collisions only ''spread'' them. We will show that this difference produces important effects in the diffusion coefficient.

The EC of the velocity components averaged over the collisional displacements, $E_{ij}^{coll}(\mathbf{x},t)$, can be calculated as the EC of the potential; the result is similar to Eqs. (31) and (32). Thus all these correlations have a diffusive evolution leading to spatially uniform functions. Consequently, one can show that $E_{ij}^{coll}(\mathbf{x},t)$ can be derived from the EC of the potential $E^{coll}(\mathbf{x},t)$ by the same equations (5) as in the collisionless case:

$$E_{xx}^{coll}=-\frac{\partial^2}{\partial y^2}E^{coll},\quad E_{yy}^{coll}=-\frac{\partial^2}{\partial x^2}E^{coll},\ \ldots\ . \tag{35}$$

In order to determine the LVC and the diffusion coefficient by means of the decorrelation trajectory method, we have to specify the spatial dependence of the EC of the potential. We consider the same model as in Ref. [9]:

$$\mathcal{E}(r)=\frac{1}{1+\dfrac{r^2}{2\lambda^2}}. \tag{36}$$

In the following calculation dimensionless variables will be used, taking as units λ for distances, $\tau_{coll}=\lambda^2/\chi$ for time, β for the potential, and $V=\beta/\lambda$ for velocities (without introducing different notations). The collision averaged EC of the potential corresponding to Eq. (36) can be written in a simpler form than the general expression (34):

$$\mathcal{E}^{coll}(r,t)=\frac{1}{2t}\int_0^\infty d\boldsymbol{\xi}\frac{\boldsymbol{\xi}\exp\left(-\dfrac{\xi^2}{4t}\right)}{\sqrt{\left(1+\dfrac{(r-\xi)^2}{2}\right)\left(1+\dfrac{(r+\xi)^2}{2}\right)}}. \tag{37}$$

The decorrelation trajectory method starts from dividing the space of realizations of the stochastic potential $\phi(\mathbf{x},t)$ in subensembles determined by condition (11). We determine the average of the doubly stochastic velocity in each subensemble. As shown in the Appendix, the result is similar to Eq. (13) obtained in the collisionless case:

$$V_j^{coll}(\mathbf{x},t;\phi^0,\mathbf{v}^0)\equiv\overline{\langle v_j(\mathbf{x}+\boldsymbol{\xi}(t),t)\rangle}\big|_{\phi^0,\mathbf{v}^0}$$
$$=v_i^0\frac{E_{ij}^{coll}(\mathbf{x},t)}{V^2}+\phi^0\frac{E_{\phi j}^{coll}(\mathbf{x},t)}{V\beta}. \tag{38}$$

Defining the x axis in the direction of the initial velocity, we have $\mathbf{v}^0=(u,0)$. The equations for the decorrelation trajectory (14) in subensemble (11) are obtained, using Eqs. (38) and (35), as

$$\frac{dX}{dt}=-Pu\frac{\partial H(X,Y,t)}{\partial Y}\exp\left(-\frac{Pt}{k}\right), \tag{39}$$

$$\frac{dY}{dt}=Pu\frac{\partial H(X,Y,t)}{\partial X}\exp\left(-\frac{Pt}{k}\right),$$

with the Hamiltonian H defined by

$$H(X,Y,t)=\left(\frac{\partial}{\partial Y}+p\right)\mathcal{E}^{coll}(R,t), \tag{40}$$

where $p=\phi^0/u$ and $R=\sqrt{X^2+Y^2}$. These equations have a Hamiltonian structure as in the collisionless case. The collisions introduce the time dependence in H in the following way. The function H [Eq. (40)] is a solution of the (*irreversible*) diffusion equation (33). This function formally acts as a Hamiltonian in Eqs. (39). There is, however, no paradox here, because $H(X,Y,t)$ is only defined for $t>0$ [the solutions of Eq. (33) form a semigroup]; hence time cannot be inversed. The seemingly ''reversible'' equations (39) thus actually describe an irreversible evolution. The decorrelation paths obtained from Eqs. (39) and (40) are much more complicated than in the collisionless case. They are no longer closed curves. The decorrelation trajectories are functions of four independent variables, $X(t)\equiv X(t;K,Pu,p)$, while in the collisionless case there were only two variables (Kut and p).

The LVC is determined according to Eq. (15) as the Lagrangian correlation observed along the decorrelation trajectories. The running diffusion coefficient obtained by integrating the LVC is

$$D(t;K,P)=\beta\frac{1}{\sqrt{2\pi}}\int_0^\infty dp\,du\,u^3$$
$$\times\exp\left(-\frac{u^2}{2}(p^2+1)\right)X(t;Pu,p,K). \tag{41}$$

This expression is similar to Eqs. (18) and (16), but due to the presence of several independent parameters it is not possible as in the collisionless case to compute $D(t)$ in the frozen turbulence limit ($K\to\infty$), thus obtaining the function F which actually determines the diffusion coefficient for any K [see Eq. (20)]. Here, the time dependent diffusion coefficient (41) must be calculated up to its asymptotic value for each value of K and P independently. The calculation procedure is much more complicated and the computation effort much larger. The only possible simplification is to determine and to store on a large size mesh r_i, t_j the collision average EC of the potential $\mathcal{E}^{coll}(r,t)$ and its first and second derivatives, and to use space and time interpolation in the numerical integration of the decorrelation trajectories. The space derivatives of $\mathcal{E}^{coll}$ which enter in the decorrelation equations (39) are calculated using the analytical derivatives of Eq. (37) and numerically performing the resulting space integrals. The order of the calculation of the two integrals in Eq. (41) is important: the integral over u has to be performed first.

The effective diffusion coefficient of collisional particles moving in a stochastic potential is obtained from Eq. (41) as

$$D_{eff}=D(K,P)+\chi, \tag{42}$$

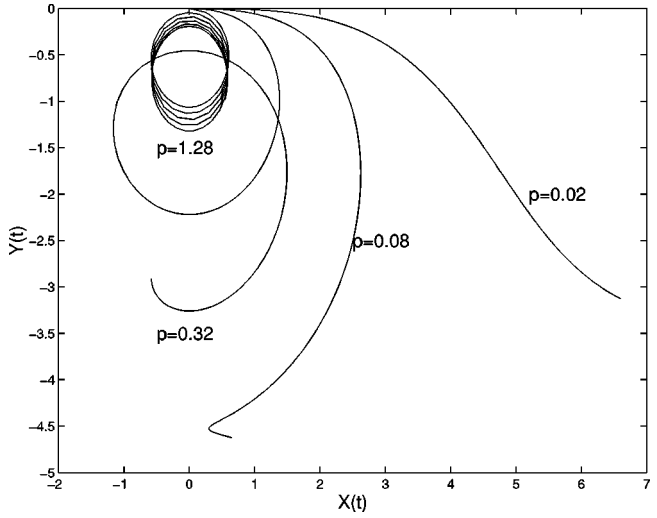

FIG. 2. Examples of decorrelation paths obtained from Eq. (39) for various values of $p \equiv \phi^0/u$; $P=50$, $K=200$, and $u=1$.

where the ''interaction'' term $D(K,P)$ is given by the asymptotic value of Eq. (41) and involves the combined action of the stochastic field and collisions. χ is the always present direct contribution of collisions.

V. RESULTS

The decorrelation paths obtained from Eq. (39) are much more complicated than in the collisionless case (a few examples are presented in Fig. 2). They show that the trapping effect is still effective at large Kubo and Péclet numbers: the trajectories corresponding to large values of $p \equiv |\phi^0/u|$ are almost closed (up to a small drift in the direction perpendicular to $\mathbf{v}^0$, here along the y axis). These trapped trajectories do not contribute to the diffusion coefficient. The latter is determined by the trajectories which are not trapped (i.e., which perform less than one rotation before they stop). They correspond to small values of p. In the collisionless case this effective range of p around $p=0$ depends on K (it goes to zero as $K \to \infty$). The collisions produce the increase of this effective range of p. Thus the collisions have a releasing effect enlarging the number of ''untrapped'' trajectories.

A nontrivial time dependence of the running diffusion coefficient is obtained from Eq. (41). As seen in Fig. 3 there appear several transient regimes at large K and P. First, at $t \ll \tau_{tr}$, there is a linear dependence $D(t) \sim t$, since the time elapsed is smaller than the transit time τ_{tr} and the particles have not yet ''seen'' the structure of the stochastic potential. Then, at $\tau_{tr} \ll t \ll \tau_{coll}$, the trapping process is effective and the diffusion coefficient is decreased. As can be observed in Fig. 3 compared to Fig. 1, this first part of the evolution of $D(t)$ is actually determined by the function $F(\theta)$ obtained in the collisionless case. Thus the collisions do not influence the evolution of the diffusion coefficient at times much smaller that the collisional characteristic time τ_{coll}. At $t \gtrsim \tau_{coll}$, a tendency of saturation of the diffusion coefficient appears.

This is similar to what happens in the collisionless problem: at $t \gtrsim \tau_c$ the time variation of the stochastic potential produces the decorrelation of the trajectories and as a result the LVC goes to zero and the diffusion coefficient saturates.

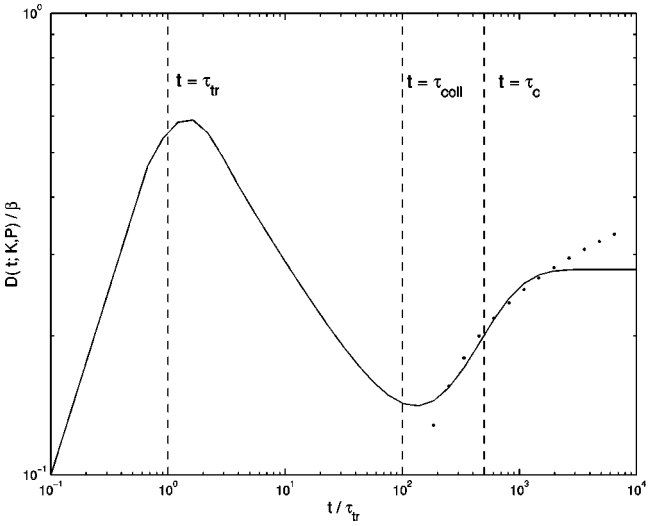

FIG. 3. The running diffusion coefficient (41) for $P=100$ and $K=500$ (continuous line) compared to the result presented in Ref. [10] (dotted line).

If the evolution of $D(t)$ would end here a result similar to Eq. (20) would be obtained: $D=\beta F(P)$. But, at $t \gg \tau_{coll}$, an increase of the diffusion coefficient is observed. After that, at $t \gtrsim \tau_c$, the time variation of the potential eventually determines the saturation of $D(t)$.

The LVC is represented in Fig. 4. Since this function has a long tail, we represent $\log(|L|)$. One can observe first the decay of the LVC in a time of the order τ_{tr}, then, between the two singularities which appear in this logarithmic representation, there is the negative LVC which is determined by particle trapping. At $t \sim \tau_{coll}$, the LVC goes to zero, and, at $t \gg \tau_{coll}$, a positive LVC appears due to collisions. This shows that the collisions do not produce the decorrelation of the trajectories but instead they ''create'' Lagrangian correlations at large times. Compared to the collisionless LVC corresponding to the same value of K and to $P \to \infty$ (dashed

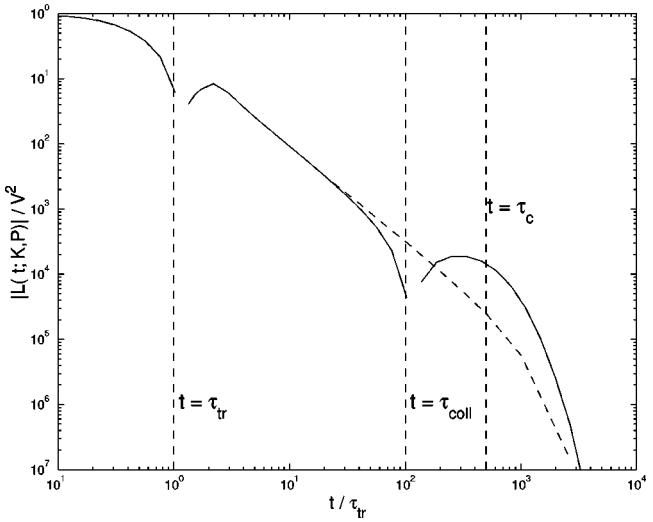

FIG. 4. The absolute value of the LVC for $P=100$ (continuous line) and for the collisionless case (dotted line); $K=500$. Between the singularities corresponding to the zeros of $L(t;K,P)$ lies the negative LVC determined by trapping. The effect of the collisions appears at large times ($t > \tau_{coll}$), where they induce a positive LVC.

line in Fig. 4), the effect of particle collisions appears very clearly: they produce a bump on the negative tail of the LVC at $\tau_{coll} \ll t \ll \tau_c$.

This picture is in agreement with the results obtained by means of the renormalization group technique which is able to determine the asymptotic scaling in t of the MSD of collisional particles moving in a *static* stochastic potential (see Refs. [10,13–15]). The heuristic argument for explaining these induced correlations is that the collisional displacement (the Brownian motion) determines long range temporal correlations in the sequence of the values of the potential encountered because in two dimensions the Brownian trajectory returns in the already visited places with probability one. As a result, the time spent during t in each visited place is of the order $\ln(t)$. A shown in Refs. [10] and [13], in a two-dimensional space, the ''interaction'' of the diffusive collisional motion with the subdiffusive process induced by the static stochastic potential determines a supradiffusive asymptotic behavior of the MSD proportional to $t\sqrt{\ln(t)}$. For the *time dependent* stochastic potential, we obtain this time dependence of the MSD as a transient regime. This can be seen in Fig. 3 where the running diffusion coefficient $D(t) \sim \sqrt{\ln(t)}$, corresponding to the above MSD, is represented by the dotted line and compared to our result $D(t;K,P)$. We note that, with the decorrelation trajectory method, we obtain this supradiffusive behavior as a transient regime in a static potential as well. This problem will be discussed in a forthcoming paper, where ''random random walks'' are studied in two and three dimensions in the case of short and long range EC of the potential.

VI. DIFFUSION REGIMES

Collisional particle diffusion in a time dependent stochastic potential depends on three characteristic times: the transit time $\tau_{tr} = \lambda / V$, the correlation time of the stochastic potential, τ_c, and the collisional time $\tau_{coll} = \lambda^2 / \chi$. They are combined in two dimensionless parameters: the Kubo number $K = \tau_c / \tau_{tr}$ and the Péclet number $P = \tau_{coll} / \tau_{tr}$. Depending on the values of these parameters, several diffusion regimes are obtained. They can be understood by analyzing the equations for the decorrelation trajectories (39).

Simple arguments show that collisions can have an important role in the interaction term of the effective diffusion coefficient (42) when $P \ll K$. In the opposite case, $P \gg K$ (or $\tau_{coll} \gg \tau_c$), during the correlation time of the stochastic potential the collisional Brownian motion is negligible compared to λ and there is no interaction between the effects of the two stochastic processes. This can easily be seen in the equation for the decorrelation trajectories (39) where the exponential factor corresponding to the time variation of the potential produces in these conditions the decay of the decorrelation velocity before the EC $\mathcal{E}^{coll}(\mathbf{x},t)$ changes significantly in time. Thus, Eqs. (39) are practically identical with those obtained in the collisionless case and the effective diffusion coefficient (42) is the sum of the diffusion coefficients determined separately by each stochastic process,

$$D_{eff} = D(K) + \chi = \beta\left(F(K) + \frac{1}{P} \right), \quad P \gg K, \tag{43}$$

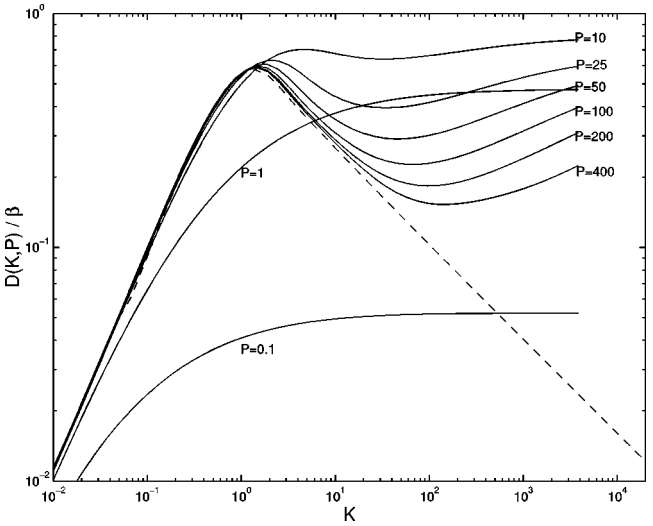

FIG. 5. The asymptotic diffusion coefficient $D(K,P)$ obtained from Eq. (41) as a function of K for several values of P.

where $D(K)$ is the collisionless diffusion coefficient deduced in Ref. [9] and given in Eq. (20).

In the quasilinear regime defined by $P \ll 1$ (and $P \ll K$ in order to have a collision effect), the interaction term of the effective diffusion coefficient $D(K,P)$ can easily be estimated from Eqs. (39). The displacement produced during the collision time τ_{coll} is much smaller than λ, and the asymptotic value of $X(t)$ is approximately cPu where the constant c is the time integral of $\partial H(X,Y,t)/\partial Y$ calculated in $X=0$ and $Y=0$ which for the EC [Eq. (36)] yields $c \cong 0.5$. Upon substitution in Eqs. (41) and (42), one obtains

$$D_{eff} = \beta\left(cP + \frac{1}{P} \right), \quad P \ll 1, \quad P \ll K. \tag{44}$$

This is a highly collisional regime ($\chi \gg \beta$) in which the stochastic potential has a small contribution: the direct contribution of collisions χ [the second term in Eq. (44)] is actually dominant. However, we note that in this strongly collisional case the scaling of the diffusion coefficient with respect to the Kubo number is of Bohm type even at $K \gg 1$, which means that the trapping process does not appear in this case (due to collisions which strongly scatter the trajectories).

The most interesting results appear in the nonlinear regime defined by $P \gg 1$, i.e., when the collisional diffusion is weak ($\chi \ll \beta$). In these conditions, the first term in Eq. (42) is dominant, the direct contribution of the collisions being negligible. When $P > 1$ and $K \gtrsim P$, the diffusion coefficient depends on both parameters.

The results are presented in Fig. 5, where the ''interaction'' term $D(K,P)/\beta$ is represented as a function of K for various values of P. First we note that the above estimates for the quasilinear regime are confirmed: the curves for $P = 0.1$ and $P = 1$ show the transition between regime (43) at $K \ll P$ to regime (44) at $K \gg P$. The transition is slow, lying on several orders of magnitude of K. This term is actually at $P \ll 1$ just a correction to the much larger collisional diffusivity χ.

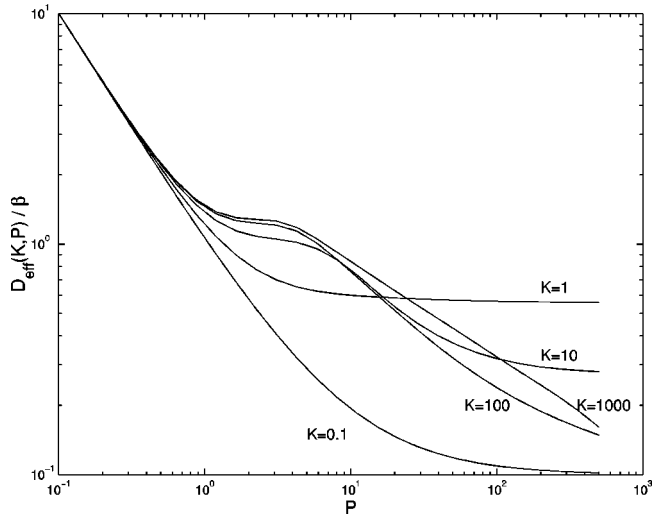

FIG. 6. The total asymptotic diffusion coefficient $D_{eff}(K,P)$ [Eq. (42)], as a function of P for several values of K.

In the nonlinear regime $P \gg 1$, we can see that an important amplification of the effective diffusion can be produced by a small collisional diffusion χ. The collisionless result (the dashed curve in Fig. 5) is also represented for comparison. As seen in Fig. 5, the scaling of the diffusion coefficient $D(K,P)$ in the Kubo number is not influenced by collisions at $K \ll P$ (where the quasilinear and the percolation regime can be observed) but after that, at $K \gtrsim P$, the K dependence is strongly changed. At the ''resonance'' condition $K \cong P$ (i.e., $\tau_c \cong \tau_{coll}$) a wide minimum of the diffusion coefficient appears. Then, as K increases, the percolation scaling is transformed into a super-Bohm regime with a K dependence stronger than linear:

$$D(K,P) \sim K^{1.17}, \quad K \gg P \gg 1. \tag{45}$$

The dependence of the effective diffusion coefficient [Eq. (42)] D_{eff}/β on the Péclet number P in presented in Fig. 6 for several values of K. One can observe that at $P \ll 1$ the effective diffusion coefficient is independent of K and can be approximated by the collisional diffusion χ. The effect of the stochastic potential becomes important at large P where the effective diffusion coefficient is much larger that χ. The contribution of the stochastic potential is independent of collisions (i.e., of P) at small K but at large K it is a complicated interaction between the dynamical trapping of the particles and the collisional Brownian motion.

VII. CONCLUSIONS

We have shown here that particle collisions can produce important modifications of the effective transport coefficients in a given two-dimensional electrostatic turbulence. We have determined the diffusion coefficient induced by the combined process of collisions and time dependent stochastic potential. A large amplification of the diffusion coefficient appears in low frequency plasma turbulence ($K \gg 1$) due to a small collisional diffusion $\chi \ll \beta$ $(P \gg 1)$. As an example, a collisional diffusion χ which is only 1% of the amplitude of the stochastic potential ($\chi = 10^{-2}\beta, P = 100$) approximately doubles the diffusion at $K = P$ (where the effect is minimum), and multiplies it by a factor of order 10 at $K \simeq 1000$. The scaling of the diffusion coefficient in the Kubo number is changed by collisions in this range of parameters: the diffusion coefficient has a minimum at $K \cong P$ and then, at $K \gg P$, a super-Bohm regime was found. The physical reason for this ''anomalous'' effect is particle trapping in the structure of the stochastic potential, combined with the collisional Brownian motion which in two dimensions generates long time correlations by returning in the places already visited.

ACKNOWLEDGMENTS

M.V. and F.S. are grateful for the support of C.E.A. and thank their colleagues in Cadarache for the very useful discussions and for the warm hospitality. This work benefitted from NATO Linkage Grant No. CRG.LG 971484 which is acknowledged.

APPENDIX

The average of the doubly stochastic velocity in the subensemble (11) [defined by Eq. (38)] can be written explicitly as

$$\mathbf{V}^{coll}(\mathbf{x},t;\phi^0,\mathbf{v}^0) = \frac{\overline{\langle \mathbf{v}\delta[\mathbf{v}-\mathbf{v}(\mathbf{x}+\boldsymbol{\xi}(t),t)]\,\delta(\mathbf{v}^0-\mathbf{v}(\mathbf{0},0))\,\delta(\phi^0-\phi(\mathbf{0},0))\rangle}}{\overline{\langle \delta(\mathbf{v}^0-\mathbf{v}(\mathbf{0},0))\,\delta(\phi^0-\phi(\mathbf{0},0))\rangle}}$$

Using the Fourier representation of the δ functions and the explicit definition of the collision average, one obtains

$$V_x^{coll} = \frac{1}{\mathcal{P}_1(\mathbf{v}^0,\phi^0)} \frac{1}{(2\pi)^5} \int\int d\mathbf{k}_1\, d\mathbf{k}_2\, dq \int d\mathbf{v}\, v_x \times \exp(-i\mathbf{k}_1\cdot\mathbf{v} - i\mathbf{k}_2\cdot\mathbf{v}^0 - iq\phi^0) \int d^2\xi\, P_1^c(\xi,t) A,$$

where

$$A \equiv \langle \exp[i\mathbf{k}_1\cdot\mathbf{v}(\mathbf{x}+\boldsymbol{\xi},t) + i\mathbf{k}_2\cdot\mathbf{v}(\mathbf{0},0) + iq\phi(\mathbf{0},0)]\rangle$$

The integral over $d\mathbf{v}$ is $(2\pi)^2 i\delta(k_{1y})(d/dk_{1x})\delta(k_{1x})$, and the integral over $d\mathbf{k}_1$ can be calculated:

$$V_x^{coll} = \frac{-(2\pi)^{-3} i}{\mathcal{P}_1(\mathbf{v}^0,\phi^0)} \int d\mathbf{k}_2\, dq \times \exp(-i\mathbf{k}_2\cdot\mathbf{v}^0 - iq\phi^0) \int d^2\xi\, P_1^c(\xi,t) \frac{\partial}{\partial k_{1x}} A|_{\mathbf{k}_1=\mathbf{0}}.$$

After calculating the average A over the realizations of the (Gaussian) stochastic potential and the derivative, one obtains

$$\frac{\partial}{\partial k_{1x}} A|_{\mathbf{k}_1=\mathbf{0}} = \left(\frac{k_{2j} E_{jx}(\mathbf{x}+\boldsymbol{\xi},t)}{V^2} + \frac{q E_{\phi x}(\mathbf{x}+\boldsymbol{\xi},t)}{V\beta} \right) \times \exp\left(-\frac{k_2^2+q^2}{2} \right).$$

The subensemble average velocity can be written as

$$V_x^{coll}=\frac{1}{\mathcal{P}_1(\mathbf{v}^0,\phi^0)}\int d^2\xi\, P_1^c(\xi,t) \times\left(\frac{E_{jx}(\mathbf{x}+\boldsymbol{\xi},t)}{V^2}\frac{\partial}{\partial v_j^0} + \frac{E_{\phi x}(\mathbf{x}+\boldsymbol{\xi},t)}{V\beta}\frac{\partial}{\partial\phi^0}\right)\mathcal{P}_1(\mathbf{v}^0,\phi^0),$$

which leads to Eq. (38). Thus, the subensemble average of the doubly stochastic velocity is similar with that obtained in the collisionless case [Eq. (13)] although the corresponding two-point probability density is not of Gaussian type. The latter is given by a convolution similar to Eq. (28):

$$\mathcal{P}_2(\mathbf{v}^0,\phi^0,\mathbf{0},0;\mathbf{v},\mathbf{x}_1,t_1) = \int d^2\xi\, P_1^c(\xi,t)\, P_2^{\phi}(\mathbf{v}^0,\phi^0,\mathbf{0},0;\mathbf{v},\mathbf{x}+\boldsymbol{\xi},t).$$

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